SOLUTIONS

1. R is reflexive as $(3, 3), (6, 6), (9, 9), (12, 12) \in R$, R is not symmetric as $(6, 12) \in R$ but $(12, 6) \notin R$, R is transitive as the only pair which needs verification is $(3, 6)$ and $(6, 12) \in R \Rightarrow (3, 12) \in R$

2. One

3. For number of one-one and onto mapping both set have equal number of elements.

   $\therefore$ Such number of mapping is zero.

4. Skew-symmetric.

$$(AB - BA)' = (AB)' - (BA)' = B'A' - A'B'$$

$$= BA - AB = -(AB - BA) \quad [\because A = A', B = B']$$

Or

(256) We have, $|A| = 4$

$$\because |adj\,(adj\,A)| = |A|^{(n-1)^2}$$

$$= 4^{(3-1)^2} = 4^4 = 256$$

5. We have, $[x, 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$

$\Rightarrow \quad [x-2 \quad 0] = 0$

$\Rightarrow \quad [x-2 \quad 0] = [0 \quad 0]$

$\therefore \quad x-2=0 \Rightarrow x=2$

6. $\begin{vmatrix} x^3 & x^2+x+1 \\ x-1 & 1 \end{vmatrix}$

$= (x^3)(1) - (x-1)(x^2+x+1)$

$= x^3 - (x^3 - 1)$

$= x^3 - x^3 + 1$

$= 1$

7. We have, $2y + x^2 = 3$

$\Rightarrow \quad 2\dfrac{dy}{dx} + 2x = 0 \Rightarrow \dfrac{dy}{dx} = -x$

$\therefore$ Slope of normal at point (1, 1) $= \dfrac{-1}{\left.\dfrac{dy}{dx}\right|_{(1,1)}} = \dfrac{-1}{-1} = 1$

Or

We have, $y = \sin x$, $\dfrac{dy}{dx} = \cos x$

$\therefore$ Slope of normal $= \left(\dfrac{-1}{\cos x}\right)_{x=0} = -1$

Hence, equation of normal at (0, 0) is

$y - 0 = -1(x-0)$

$\therefore \quad x + y = 0$

8. We have, $\quad y = x^{\sin x}$

$\Rightarrow \quad \log y = \sin x \log$

$\Rightarrow \quad \dfrac{1}{y}\dfrac{dy}{dx} = \cos x \log x + \dfrac{\sin x}{x}$

$\Rightarrow \quad \dfrac{dy}{dx} = y\left(\cos x \log x + \dfrac{\sin x}{x}\right)$

$\Rightarrow \quad \dfrac{dy}{dx} = x^{\sin x}\left(\cos x \log x + \dfrac{\sin x}{x}\right)$

Or

We have,

$y = \sqrt{a^2 - x^2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{a^2-x^2}}(0-2x) = \dfrac{-x}{\sqrt{a^2-x^2}}$

$\Rightarrow \quad \sqrt{a^2-x^2}\dfrac{dy}{dx} + x = 0$

$\Rightarrow \quad y\dfrac{dy}{dx} + x = 0 \qquad [\because y = \sqrt{a^2-x^2}]$

Hence proved.

9. Let $I = \displaystyle\int_2^3 \frac{dx}{1-x^2}$

$I = \dfrac{1}{2}\left[\log\left|\dfrac{1+x}{1-x}\right|\right]_2^3$

$I = \dfrac{1}{2}\left[\log\dfrac{4}{2} - \log\dfrac{3}{1}\right] \quad \therefore \; I = \dfrac{1}{2}\log\dfrac{2}{3}$

Or Let $I = \displaystyle\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$

$I = [\sec^{-1} x]_1^2 = \sec^{-1} 2 - \sec^{-1} 1$

$I = \dfrac{\pi}{3} - 0 = \dfrac{\pi}{3}$

10. Let $\vec{b}_1 = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b}_2 = -\hat{i} + 3\hat{j} + 4\hat{k}$

Then, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix}$

$= \hat{i}(5) - \hat{j}(5) + \hat{k}(5) = \vec{c}$ (say)

$\Rightarrow \quad |\vec{b}_1 \times \vec{b}_2| = \sqrt{(5)^2 + (5)^2 + (5)^2} = 5\sqrt{3}$

Now, $\quad \hat{c} = \dfrac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|} = \dfrac{\hat{i}}{\sqrt{3}} - \dfrac{\hat{j}}{\sqrt{3}} + \dfrac{\hat{k}}{\sqrt{3}}$

Hence, required vector $= \pm (10\sqrt{3}\,\hat{c})$

$= \pm (10\hat{i} - 10\hat{j} + 10\hat{k})$

11. We have, $|\vec{a}| = 1$, $\vec{a} \times \hat{i} = \hat{j}$

$\therefore \quad \vec{a} = \hat{k}$

So, $\quad \vec{a} \cdot \hat{i} = \hat{k} \cdot \hat{i} = 0$

12. We know that, $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

So, angle between $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ is π.

13. We have, $\vec{a} \cdot \vec{b} = 6$, $|\vec{a}| = 3$ and $|\vec{b}| = 4$

$\therefore$ Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{6}{4} = \dfrac{3}{2}$

14. We have, $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$

$\Rightarrow \quad |\vec{a}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow \quad |\vec{a}|^2 = 8 + 1 \qquad [\because |\vec{b}| = 1]$

$\Rightarrow \quad |\vec{a}|^2 = 9$

$\Rightarrow \quad |\vec{a}| = 3$

15. Here, $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.5$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 0.5 = 0.4 + 0.3 - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = 0.7 - 0.5 = 0.2$

Now, $P(B' \cap A) = P(A) - P(A \cap B)$

$= 0.4 - 0.2 = 0.2 = \dfrac{2}{10} = \dfrac{1}{5}$

16. We have, A and B are independent event.

$\therefore \quad P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow \quad P(B) = \dfrac{P(A \cap B)}{P(A)}$

$= \dfrac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} = \dfrac{1}{2}$

Or

We have, $P(A) = 0.3$, $P(B) = 0.6$, $P\left(\dfrac{B}{A}\right) = 0.5$

Now, $P\left(\dfrac{B}{A}\right) = \dfrac{P(B \cap A)}{P(A)}$

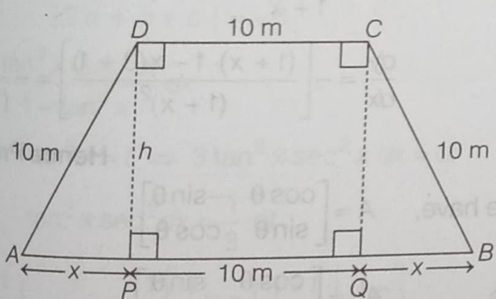
$\Rightarrow \quad P(B \cap A) = 0.3 \times 0.5 = 0.15$

Again, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.6 - 0.15$

$= 0.9 - 0.15 = 0.75$

17.



(i) (b) In ΔAPD,

We have $\quad AP^2 + PD^2 = AD^2$

$\Rightarrow \quad x^2 + h^2 = (10)^2$

$\Rightarrow \quad x^2 + h^2 = 100$

(ii) (c) We have, $x^2 + h^2 = 100$

$\Rightarrow \quad h = \sqrt{100 - x^2}$

Now, Area of gate = Area of Trapezium $ABCD$

$\Rightarrow \quad A = \dfrac{1}{2}(AB + CD) \times DP$

$= \dfrac{1}{2}(10 + 10 + 2x) \times \sqrt{100 - x^2}$

$= (10 + x)\sqrt{100 - x^2}$

(iii) (a) We have, $A = (10 + x)\sqrt{100 - x^2}$

$\Rightarrow \quad \dfrac{dA}{dx} = \sqrt{100 - x^2} - \dfrac{x(10 + x)}{\sqrt{100 - x^2}}$

$= \dfrac{100 - 10x - 2x^2}{\sqrt{100 - x^2}}$

For maxima, $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad \dfrac{100 - 10x - 2x^2}{\sqrt{100 - x^2}} = 0$

$\Rightarrow \quad 100 - 10x - 2x^2 = 0$

$\Rightarrow \quad x^2 + 5x - 50 = 0$

$\Rightarrow \quad (x + 10)(x - 5) = 0$

$\Rightarrow \quad x = 5 \quad [\because x > 0 \therefore x + 10 \neq 0]$

Again,

$$\dfrac{d^2A}{dx^2} = \dfrac{\sqrt{100 - x^2}(-10 - 4x) + \dfrac{(100 - 10x - 2x^2)x}{\sqrt{100 - x^2}}}{100 - x^2}$$

$$= \dfrac{2x^3 - 300x - 1000}{(100 - x^2)^{\frac{3}{2}}}$$

$\Rightarrow \quad \left.\dfrac{d^2A}{dx^2}\right|_{x=5} = -\dfrac{30}{\sqrt{75}} < 0$

So, A is maximum at $x = 5$ m

(iv) (b) We have,

$h = \sqrt{100 - x^2}$

$= \sqrt{100 - 25}$

$= \sqrt{75}$

$= 5\sqrt{3}$ m

(v) (a) Maximum value of $A = \dfrac{1}{2}(10 + 5)\sqrt{100 - 25}$

$= \dfrac{75\sqrt{3}}{2}$ m^2

18. (i) (c) We know that,

$\Sigma P(X) = 1$

$\Rightarrow \quad 3C^3 + 4C - 10C^2 + 5C - 1 = 1$

$\Rightarrow \quad 3C^3 - 10C^2 + 9C - 2 = 0$

(ii) (b) We have,

$3C^3 - 10C^2 + 9C - 2 = 0$

$\Rightarrow \quad (3C - 1)(C^2 - 3C + 2) = 0$

$\Rightarrow \quad C = \dfrac{1}{3}$

(iii) (b) $P(X < 2) = P(X = 0) + P(X = 1)$

$= 3C^3 + 4C - 10C^2$

$$= 3\left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right) - 10\left(\frac{1}{3}\right)^2$$

$$= \frac{1}{9} + \frac{4}{3} - \frac{10}{9}$$

$$= \frac{1+12-10}{9} = \frac{3}{9} = \frac{1}{3}$$

(iv) (a) $P(X=1) = 4C - 10C^2$

$$= \frac{4}{3} - \frac{10}{9}$$

$$= \frac{2}{9}$$

(V) (d) $P(X \geq 0) = P(X=0) + P(X=1) + P(X=2) = 1$

19. Clearly, all positive real numbers have the same image equal to 1. So, f is many-one function.

We observe that the range of f is $\{-1, 0, 1\}$ which is not equal to the codomain of f. So, f is not onto.

Hence, f is neither one-one nor onto. (2)

Or

For R to be reflexive (b, b) and (c, c) should belong to R and for R to be transitive (a, c) should belong to R as $(a, b) \in R$ and $(b, c) \in R$. Hence, minimum number of ordered pairs to be added is R is 3. (2)

20. Let OA is inclined at an angle Y to OZ.

Given, $\alpha = 60^\circ$ and $\beta = 45^\circ$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2 60^\circ + \cos^2 45^\circ + \cos^2\gamma = 1$$

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{2} + \cos^2\gamma = 1 \Rightarrow \cos^2\gamma = \frac{1}{4}$$

$$\Rightarrow \quad \cos\gamma = \frac{1}{2} \Rightarrow \gamma = 60^\circ \quad (1)$$

$$OA = |OA|\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

$$= 10\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

$$\vec{OA} = 5\hat{i} + 5\sqrt{2}\hat{i} + 5\hat{k} \quad (1)$$

21. We have,

$$f(x) = \frac{x-2}{x+1}, x \neq -1$$

$$f'(x) = \frac{(x+1)(1) - (x-2)(1)}{(x+1)^2}, x \neq -1$$

$$= \frac{3}{(x+1)^2}, x \neq -1 \quad (1)$$

Clearly, $f'(x) = \frac{3}{(x+1)^2} > 0$, for all $x \in R - \{-1\}$.

So, $f(x)$ is increasing on $R - \{-1\}$. (1)

22. We have,

$$\sec\left(\frac{x+y}{x-y}\right) = a$$

$$\Rightarrow \quad \frac{x+y}{x-y} = \sec^{-1} a \quad (1)$$

On differentiating both sides, we get

$$\frac{\left(1+\frac{dy}{dx}\right)(x-y) - (x+y)\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow \quad x - y + (x-y)\frac{dy}{dx} - x - y + (x+y)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x - y + x + y)\frac{dy}{dx} = 2y \Rightarrow \quad 2x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$ **Hence proved.** (1)

Or

Given, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow \quad x^2(1+y) = y^2(1+x)$ [squaring both sides]

$\Rightarrow \quad x^2 - y^2 = y^2x - x^2y$

$\Rightarrow \quad (x+y)(x-y) = -xy(x-y) \Rightarrow x+y = -xy$ $[\because x \neq y]$

$\Rightarrow \quad x = -y - xy = y(1+x) = -x$ (1)

$\Rightarrow \quad y = -\frac{x}{1+x}$

$$\therefore \quad \frac{dy}{dx} = -\left[\frac{(1+x)\cdot 1 - x(0+1)}{(1+x)^2}\right] = -\frac{1}{(1+x)^2}$$

Hence Proved. (1)

23. We have, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\Rightarrow \quad A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (1/2)$$

Now, $A^T + A = I$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2\cos\theta = 1 \quad (1/2)$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z \quad (1)$$

24. Let A be the event of drawing a diamond card in the first draw and B be the event of drawn a diamond card in the second draw. Then,

$$P(A) = \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52} = \frac{1}{4}$$ (1/2)

After drawing a diamond card in first draw 51 cards are left out of which 12 cards are diamond cards.

and $P(B/A)$ = Probability of drawing a diamond card in second draw when a diamond card has already been drawn in first draw

$$= \frac{^{12}C_1}{^{51}C_1} = \frac{12}{51} = \frac{4}{17}$$ (1/2)

$\therefore$ Required probability $= P(A \cap B)$

$$= P(A)\,P(B/A)$$
$$= \frac{1}{4} \times \frac{4}{17} = \frac{1}{17}$$ (1)

25. We have, $\vec{a} \perp \vec{b} \perp \vec{c}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\therefore$ $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ (1)

and $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

Now, $|2\vec{a} + \vec{b} + \vec{c}|^2 = (2\vec{a} + \vec{b} + \vec{c})\cdot(2\vec{a} + \vec{b} + \vec{c})$

$$= 4|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$$
$$= 4 + 1 + 1 = 6$$

$\therefore$ $|2\vec{a} + \vec{b} + \vec{c}| = \sqrt{6}$ (1)

26. $\int \frac{\tan^2 x \cdot \sec^2 x}{1 - \tan^6 x}\, dx$

Let $\tan^3 x = t \Rightarrow 3\tan^2 x \sec^2 x\, dx = dt$

$\tan^2 x \sec^2 dx = \frac{1}{3}\cdot dt$ (1)

$\therefore \frac{1}{3}\int \frac{dt}{1 - t^2} = \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C\right]$

$\left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$

$= \frac{1}{6}\log\left|\frac{1+t}{1-t}\right| + C$

Now, put the value of t, we get

$\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x}\, dx = \frac{1}{6}\log\left|\frac{1 + \tan^3 x}{1 - \tan^3 x}\right| + C$ (1)

Or $\int \sin x \cdot \log \cos x\, dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\therefore -\int \log t\, dt \Rightarrow -\int (\log t)\cdot 1 dt$

$\Rightarrow -\left[\log t \int 1dt - \int \left\{\frac{d}{dt}(\log t)\int 1dt\right\}dt\right]$ (1)

$\Rightarrow -\left[(\log t)\cdot t - \int \frac{1}{t}\cdot t\, dt\right] \Rightarrow -[t\cdot\log t - \int 1dt]$

$\Rightarrow -[t\cdot\log t - t] + C$

$\Rightarrow -t\cdot\log t + t + C$

$\Rightarrow -\cos x \log\cos x + \cos x + C$ (1)

27. Let $I = \int_0^{\frac{\pi}{2}} \log\tan x\, dx$ ...(i)

$I = \int_0^{\frac{\pi}{2}} \log\tan\left(\frac{\pi}{2} - x\right)dx \quad \left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$I = \int_0^{\frac{\pi}{2}} \log\cot x\, dx$ ...(ii) (1)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} (\log\tan x + \log\cot x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \log(\tan x \cot x)\,dx = \int_0^{\frac{\pi}{2}} \log 1\, dx = 0$$

$\therefore$ $I = 0$ (1)

28. We have, $(1 + x^2)\frac{dy}{dx} + 2xy = \cot x$

$\Rightarrow \frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{\cot x}{1 + x^2}$

Which is a linear differential equation.

$\therefore$ $\text{IF} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$ (1)

The solution is given by

$$y\cdot(1 + x^2) = \int \cot x\, dx + c$$
$$y(1 + x^2) = \log\sin x + c$$ (1)

29. Let $\cos^{-1}x = \alpha, \cos^{-1}y = \beta$ and $\cos^{-1}z = \gamma$

$\cos\alpha = x, \cos\beta = y$ and $\cos\gamma = z$

Since, $\alpha + \beta + \gamma = \pi$

$\therefore$ $\alpha + \beta = \pi - \gamma$ (1)

Now, $\cos(\alpha + \beta) = \cos(\pi - \gamma)$

$\Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\cos\gamma$

$\Rightarrow xy - \sqrt{1 - x^2}\sqrt{1 - y^2} = -z$

$\Rightarrow xy + z = \sqrt{1 - x^2}\sqrt{1 - y^2}$

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

[squaring on both sides]

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$ **Hence proved.** (2)

Or

We know that domain of $\cos^{-1}x$ is $[-1, 1]$.

So, $f(x) = \cos^{-1}(x^2 - 4)$ will be defined.

When $\quad -1 \le x^2 - 4 \le 1$

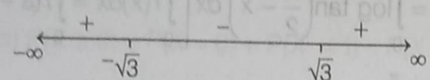
$\Rightarrow \quad -1 + 4 \le x^2 - 4 + 4 \le 1 + 4$ (1)

$\Rightarrow \quad 3 \le x^2 \le 5$

Now, when $x^2 \ge 3$

$\Rightarrow \quad x^2 - 3 \ge 0$

$\Rightarrow \quad (x - \sqrt{3})(x + \sqrt{3}) \ge 0$

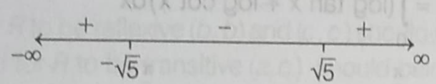


$\therefore \quad x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$ ...(i)

when $\quad x^2 \le 5$

$\Rightarrow \quad x^2 - 5 \le 0$ (1)

$\Rightarrow \quad (x - \sqrt{5})(x + \sqrt{5}) \le 0$



$\therefore \quad x \in [-\sqrt{5}, \sqrt{5}]$ ...(ii)

From Eqs. (i) and (ii), we get

$x \in [-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$ (1)

30. Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right) dx$

$\left[\because \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta} \right]$

$= \int \frac{\sin x + \cos x}{\sqrt{\sin x}\sqrt{\cos x}}\, dx$ (1)

$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\, dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x)\, dx = dt$ (1)

$\therefore \quad I = \sqrt{2} \int \frac{1}{\sqrt{1 - t^2}}\, dt$

$= \sqrt{2} \sin^{-1}(t) + c \left[\because \int \frac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1} x + c \right]$

$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + c \quad [\because t = \sin x - \cos x]$

(1)

Or

Let $\quad I = \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$

$= \int e^x \left[\frac{1 + 2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} \right] dx$ (1)

$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \right]$

$= \int e^x \left[\frac{1}{2\cos^2(x/2)} + \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} \right] dx$

$= \int e^x \left[\frac{1}{2}\sec^2\left(\frac{x}{2}\right) + \tan\frac{x}{2} \right] dx$

Thus, we have integration of the form

$\int e^x [f(x) + f'(x)]\, dx$ (1)

Here, $\quad f(x) = \tan\frac{x}{2} \Rightarrow f'(x) = \sec^2\frac{x}{2} \cdot \frac{1}{2} = \frac{1}{2}\sec^2\frac{x}{2}$

$\therefore \quad I = e^x f(x) + c = e^x \tan\frac{x}{2} + c$

$\left[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c \right]$ (1)

31. We have,

$(x^3 - 3xy^2)\, dx = (y^3 - 3x^2 y)\, dy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y}$

Clearly, the given equation is a homogeneous equation. (1/2)

Put $\quad y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$v + x\frac{dv}{dx} = \frac{x^3 - 3x^3 v^2}{v^3 x^3 - 3x^3 v}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - v^4}{v^3 - 3v}$

$\Rightarrow \quad \int \frac{v^3 - 3v}{1 - v^4}\, dv = \int \frac{1}{x}\, dx$

$\Rightarrow \int \frac{v^3}{1 - v^4}\, dv - 3\int \frac{v}{1 - v^4}\, dv = \log|x| + \log c$ (1/2)

$\Rightarrow \frac{-1}{4} \int \frac{-4v^3}{1 - v^4}\, dv - \frac{3}{2} \int \frac{2v}{1 - (v^2)^2}\, dv = \log|cx|$

$\Rightarrow -\frac{1}{4} \int \frac{dt_1}{t_1} - \frac{3}{2} \int \frac{dt_2}{1 - (t_2)^2} = \log|cx|$

where, $t_1 = 1 - v^4$ in first integral and $t_2 = v^2$ in second integral.

$\Rightarrow -\frac{1}{4}\log|1 - v^4| - \frac{3}{2} \times \frac{1}{2 \times 1} \log\left| \frac{1 + t}{1 - t} \right| = \log|cx|$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log|cx|$$

$$\Rightarrow \quad -\log|1-v^4| - 3\log\left|\frac{1+v^2}{1-v^2}\right| = 4\log|cx|$$

(1)

$$\Rightarrow \quad \log\left|(1-v^4)^{-1}\left(\frac{1+v^2}{1-v^2}\right)^{-3}\right| = \log|(cx)^4|$$

$$\Rightarrow \quad \frac{1}{1-v^4} \times \left(\frac{1-v^2}{1+v^2}\right)^3 = (cx)^4$$

$$\Rightarrow \quad (1-v^2)^2 = (1+v^2)^4 (cx)^4$$

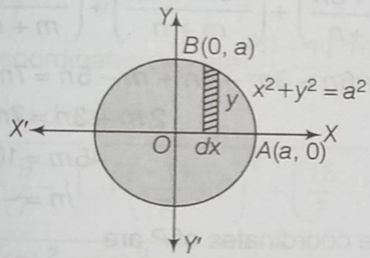
$$\Rightarrow \quad 1-v^2 = (1+v^2)^2 (cx)^2$$

$$\Rightarrow \quad 1-\frac{y^2}{x^2} = \left(1+\frac{y^2}{x^2}\right)^2 (cx)^2 \qquad [\because v = y/x]$$

$$\Rightarrow \quad x^2 - y^2 = (x^2+y^2)^2 c^2$$

(1)

32. Given equation of circle is $x^2 + y^2 = a^2$, its centre is (0, 0) and radius is a. It cuts the X-axis at $A(a, 0)$ and Y-axis at $B(0, a)$. Also, it is symmetrical about X and Y-axes both.



(1)

Clearly, area of region in I quadrant

$$= \int_0^a y\,dx = \int_0^a \sqrt{a^2 - x^2}\,dx \text{ [consider vertical strip]}$$

$[\because x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$, as y is in I quadrant$]$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2}\sin^{-1} 1\right) - 0\right] = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

(1)

$$\left[\because \sin^{-1}(1) = \frac{\pi}{2}\right]$$

Now, required area $= 4 \times$ Area of region in I quadrant

$$= 4 \times \frac{\pi a^2}{4} = \pi a^2 \text{ sq units}$$

[since, region is symmetrical in all quadrants] (1)

33. Given, equation of curve is

$$y = \frac{x-7}{(x-2)(x-3)} \qquad \ldots(i)$$

To find the intersection of given curve with X-axis.

Put $y = 0$ in Eq. (i), we get

$$0 = \frac{x-7}{(x-2)(x-3)}$$

(1)

$$\Rightarrow \quad x - 7 = 0 \Rightarrow x = 7$$

Thus, the curve cut the X-axis at (7, 0).

Now, differentiating equation of curve w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x-2)(x-3)\cdot 1 - (x-7)[(x-2)\cdot 1 + (x-3)\cdot 1]}{[(x-2)(x-3)]^2}$$

$$= \frac{(x-2)(x-3) - (x-7)(2x-5)}{[(x-2)(x-3)]^2}$$

$$= \frac{(x-2)(x-3)\left[1 - \frac{(x-7)}{(x-2)(x-3)}(2x-5)\right]}{[(x-2)(x-3)]^2}$$

$$= \frac{1 - y(2x-5)}{(x-2)(x-3)} \qquad \text{[using Eq. (i)]}$$

(1)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(7,0)} = \frac{1-0}{5\cdot 4} = \frac{1}{20}$$

Thus, slope of tangent $= \frac{1}{20}$

and slope of normal $= \frac{-1}{\text{slope of tangent}} = -20$.

Hence, the equation of tangent at (7, 0) is

$$y - 0 = \frac{1}{20}(x-7) \Rightarrow 20y - x + 7 = 0$$

and the equation of normal at (7, 0) is $y - 0 = -20(x-7)$

or $20x + y - 140 = 0$.

(1)

34. Given, $x = 3\cos\theta - \cos^3\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = -3\sin\theta - 3\cos^2\theta(-\sin\theta)$$

$$= 3\cos^2\theta\sin\theta - 3\sin\theta$$

$$= 3\sin\theta(\cos^2\theta - 1) \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= -3\sin^3\theta \qquad \ldots(i)$$

and $\quad y = 3\sin\theta - \sin^3\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

$$= 3\cos\theta(1 - \sin^2\theta)$$

$$= 3\cos\theta\cdot\cos^2\theta = 3\cos^3\theta \qquad \ldots(ii)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos^3\theta}{-3\sin^3\theta} = -\frac{\cos^3\theta}{\sin^3\theta}$$

(1)

Now, equation of the normal at $(3\cos\theta - \cos^3\theta, 3\sin\theta - \sin^3\theta)$ is

$$y - 3\sin\theta + \sin^3\theta = -\frac{-1}{\frac{\cos^3\theta}{\sin^3\theta}}(x - 3\cos\theta + \cos^3\theta)$$

$$\left[\text{equation of normal is } y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)}}(x - x_1)\right]$$

$$\Rightarrow y - 3\sin\theta + \sin^3\theta = \frac{\sin^3\theta}{\cos^3\theta}(x - 3\cos\theta + \cos^3\theta)$$

$$\Rightarrow y\cos^3\theta - 3\sin\theta\cos^3\theta + \sin^3\theta\cos^3\theta = x\sin^3\theta - 3\sin^3\theta\cos\theta + \sin^3\theta\cos^3\theta \quad (1)$$

$$\Rightarrow y\cos^3\theta - x\sin^3\theta + 3(\sin^3\theta\cos\theta - \sin\theta\cos^3\theta) = 0$$

$$\Rightarrow y\cos^3\theta - x\sin^3\theta + 3\sin\theta\cos\theta(\sin^2\theta - \cos^2\theta) = 0$$

$$\Rightarrow y\cos^3\theta - x\sin^3\theta - \frac{3}{2}\sin 2\theta\cos 2\theta = 0$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta, \cos^2\theta - \sin^2\theta = \cos 2\theta]$

$$\therefore y\cos^3\theta - x\sin^3\theta - \frac{3}{4}\sin 4\theta = 0$$

$[\because \sin 4\theta = 2\sin 2\theta\cos 2\theta]$

$\therefore 4(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta$ **Hence proved.** (1)

35. We have, $y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$ ...(i)

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\cos^{-1}x}{\sqrt{1-x^2}}\right) - \frac{d}{dx}(\log\sqrt{1-x^2}) \quad (1/2)$$

$$= \frac{\sqrt{1-x^2}\left[x\cdot\frac{(-1)}{\sqrt{1-x^2}} + \cos^{-1}x\right] - x\cos^{-1}x\cdot\frac{1}{2\sqrt{1-x^2}}(-2x)}{(\sqrt{1-x^2})^2} - \frac{1}{\sqrt{1-x^2}}\cdot\frac{(-2x)}{2\sqrt{1-x^2}} \quad (1/2)$$

$$= \frac{-x + \sqrt{1-x^2}\cos^{-1}x + \frac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2} + \frac{x}{(\sqrt{1-x^2})^2} \quad (1)$$

$$= \frac{-x + \sqrt{1-x^2}\cos^{-1}x + \frac{x^2\cos^{-1}x}{\sqrt{1-x^2}} + x}{(\sqrt{1-x^2})^2}$$

$$= \frac{(1-x^2)\cos^{-1}x + x^2\cos^{-1}x}{(\sqrt{1-x^2})^3} = \frac{\cos^{-1}x}{(1-x^2)^{3/2}} \quad (1)$$

Hence proved.

36. The equation of a plane passing through the point $L(2, 2, 1)$ is

$$a(x-2) + b(y-2) + c(z-1) = 0 \quad \text{...(i) (1/2)}$$

Also, it is passes through the points $M(3, 0, 1)$ and $N(4, -1, 0)$, respectively.

$\therefore a(3-2) + b(0-2) + c(1-1) = 0$

$\Rightarrow a - 2b = 0$

$\Rightarrow a = 2b$ ...(ii) (1/2)

and $a(4-2) + b(-1-2) + c(0-1) = 0$

$\Rightarrow 2a - 3b - c = 0$

$\Rightarrow 2(2b) - 3b - c = 0$ [from Eq. (ii)]

$\Rightarrow b - c = 0$

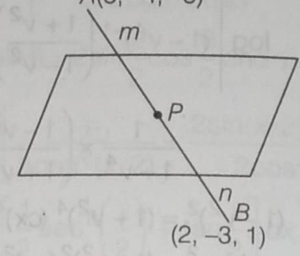
$\Rightarrow c = b$

On putting $a = 2b$ and $c = b$ in Eq. (i), we get

$2b(x-2) + b(y-2) + b(z-1) = 0$

$\Rightarrow 2x - 4 + y - 2 + z - 1 = 0$ [divide by b]

$\Rightarrow 2x + y + z = 7$ ...(iii) (1)



Let the point P divide the line joining points A and B in the ratio $m : n$.

Then, coordinates of P are

$$P\left(\frac{2m+3n}{m+n}, \frac{-3m-4n}{m+n}, \frac{m-5n}{m+n}\right). \quad (1)$$

Since, the line crosses the plane at point P. So, the coordinates of point P satisfy the equation of plane $2x + y + z = 7$.

$$\therefore 2\left(\frac{2m+3n}{m+n}\right) + \left(\frac{-3m-4n}{m+n}\right) + \left(\frac{m-5n}{m+n}\right) = 7$$

$\Rightarrow 4m + 6n - 3m - 4n + m - 5n = 7m + 7n$

$\Rightarrow 2m - 3n = 7m + 7n$

$\Rightarrow -5m = 10n$

$\Rightarrow m = -2n$ ...(iv) (1)

Now, the coordinates of P are

$$\left[\frac{2\times(-2n)+3n}{-2n+n}, \frac{-3\times(-2n)-4n}{-2n+n}, \frac{-2n-5n}{-2n+n}\right],$$

i.e. $\left(\dfrac{-n}{-n}, \dfrac{2n}{-n}, \dfrac{-7n}{-n}\right)$ or $(1, -2, 7)$

From Eq. (iv),

$$m = -2n \Rightarrow \frac{m}{n} = -2$$

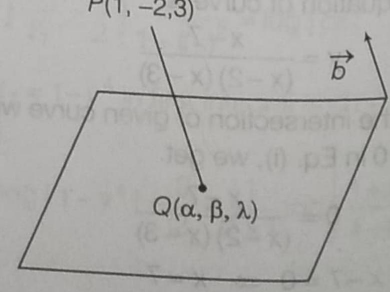
Hence, P divides the line joining points A and B externally in the ratio 2 :1. (1)

Or

Let $P(1, -2, 3)$ be the given point and $Q(\alpha, \beta, \gamma)$ be the point on the given plane

$$x - y + z = 5 \quad \text{...(i)}$$

Such that PQ is parallel to given line whose direction ratios are $(2, 3, -6)$.



(1)

Now, $\vec{PQ}$ = Position vector of Q − Position vector of P

$$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})$$
$$= (\alpha - 1)\hat{i} + (\beta + 2)\hat{j} + (\gamma - 3)\hat{k}$$

Since, $\vec{PQ}$ is parallel to $\vec{b}$.

$$\therefore \quad \frac{\alpha - 1}{2} = \frac{\beta + 2}{3} = \frac{\gamma - 3}{-6} = \lambda \text{ (say)}$$

$\Rightarrow \quad \alpha = 2\lambda + 1, \beta = 3\lambda - 2 \text{ and } \gamma = -6\lambda + 3$ ...(ii) (1)

Since, the point $Q(\alpha, \beta, \gamma)$ lies on the plane (i), so it satisfies.

$$\therefore \quad \alpha - \beta + \gamma = 5$$
$$\Rightarrow (2\lambda + 1) - (3\lambda - 2) + (-6\lambda + 3) = 5$$
$$\Rightarrow \quad -7\lambda + 6 = 5 \Rightarrow \lambda = \frac{1}{7}$$ (1)

Now, put $\lambda = \frac{1}{7}$ in Eq. (ii), we get

$$\alpha = 2 \times \frac{1}{7} + 1, \beta = 3 \times \frac{1}{7} - 2 \text{ and } \gamma = -6 \times \frac{1}{7} + 3$$
$$\Rightarrow \alpha = \frac{9}{7}, \beta = \frac{-11}{7} \text{ and } \gamma = \frac{15}{7}$$

Hence, coordinates of Q are $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$. (1)

∴ Required distance

$$PQ = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(\frac{-11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2}$$
$$= \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2}$$
$$= \frac{1}{7}\sqrt{4 + 9 + 36} = \frac{7}{7} = 1 \text{ unit}$$ (1)

37. The linear programming problem is

Maximise $Z = 0.08x + 0.10y$

Subject to the constraints

$$x + y \le 12000, \; x \ge 2000, \; y \ge 4000$$

and $\quad x \ge 0, y \ge 0$

Consider the constraints as equations, we get

$$x + y = 12000 \quad \text{...(i)}$$
$$x = 2000 \quad \text{...(ii)}$$
$$y = 4000 \quad \text{...(iii)}$$

and $\quad x, y = 0$

Table for $x + y = 12000$ is (1)

x	0	12000
y	12000	0

So, line passes through the points (0, 12000) and (12000,0).

On putting (0, 0) in the inequality $x + y \le 12000$, we get

$\Rightarrow \quad 0 + 0 \le 12000$

$\quad 0 \le 12000$ [true]

∴ The shaded region is towards the origin. (1)

∵ Line $x = 2000$ is parallel to Y-axis.

On putting (1000,0) in the inequality $x \ge 2000$, we get $1000 \ge 2000$ [false]

∴ The shaded region is at the right side of the line.

∵ Line $y = 4000$ is parallel to X-axis.

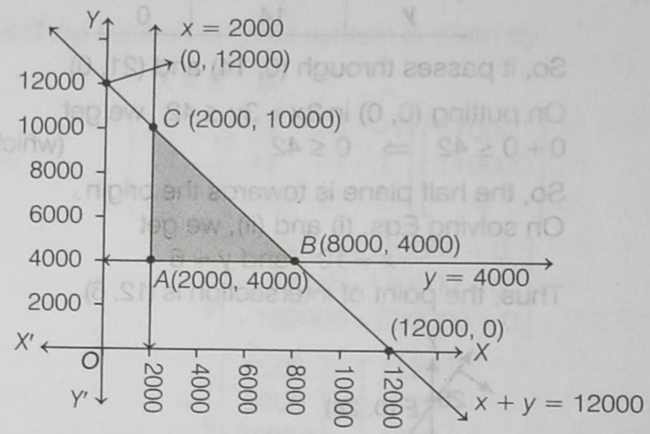
On putting (0,6000) in the inequality $y \ge 4000$, we get $6000 \ge 4000$ [true]

∴ The shaded region is above the line.

The intersection point of lines (ii) and (iii), (i) and (iii), (i) and (ii) are respectively,

$A(2000, 4000)$, $B(8000, 4000)$ and $C(2000, 10000)$. (1)

Now, plot the graph of the system of inequalities. The shaded portion ABC represents the feasible region which is bounded.



(1)

And the coordinates of the corner points are $A(2000, 4000)$, $B(8000, 4000)$ and $C(2000, 10000)$, respectively.

Now, the values of Z at each corner point are given below

Corner points	$Z = 0.08x + 0.10y$
$A(2000, 4000)$	$Z = 0.08(2000) + 0.10(4000) = 160 + 400 = 560$
$B(8000, 4000)$	$Z = 0.08(8000) + 0.10(4000) = 640 + 400 = 1040$
$C(2000, 10000)$	$Z = 0.08(2000) + 0.10(10000) = 160 + 1000 = 1160$ (maximum)

∴ Maximum value of Z is 1160 at (2000, 10000). (1)

Or

The given LPP is maximise

$$Z = x + y$$

Or

Subject to constraints

$$3x + 2y \le 48$$
$$2x + 3y \le 42$$
$$x, y \ge 0$$

and (1)

On considering the inequalities as equations, we get

$3x + 2y = 48$ ...(i)

$2x + 3y = 42$ ...(ii)

Table for line $3x + 2y = 48$ is

x	0	16
y	24	0

So, it passes through (0, 24) and (16, 0).

On putting (0, 0) in $3x + 2y \le 48$, we get

$$0 + 0 \le 48$$

$\Rightarrow$ $0 \le 48$ (which is true)

So, the half plane is towards the origin. (1½)

Table for $2x + 3y = 42$ is

x	0	21
y	14	0

So, it passes through (0, 14) and (21, 0).

On putting (0, 0) in $2x + 3y \le 42$, we get

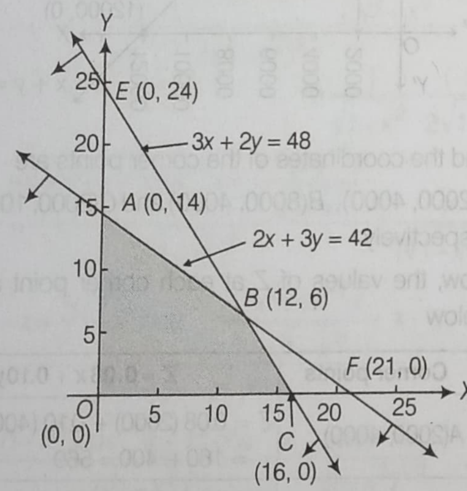
$0 + 0 \le 42 \Rightarrow 0 \le 42$ (which is true)

So, the half plane is towards the origin.

On solving Eqs. (i) and (ii), we get

$$x = 12 \text{ and } y = 6$$

Thus, the point of intersection is (12, 6). (1)



From the graph, $OABCD$ is the feasible region which is bounded. The corner points are $O(0, 0)$, $A(0, 14)$ $B(12, 6)$ and $C(16, 0)$.

The values of Z at corner points are as follows.

Corner points	Value of $Z = x + y$
$O(0, 0)$	Value of $Z = x + y$
$A(0, 14)$	$Z = 0 + 14 = 14$
$B(12, 6)$	$Z = 12 + 6 = 18$ (maximum)
$C(16, 0)$	$Z = 16 + 0 = 16$

From the table, the maximum value of Z is 18 at $B(12, 6)$. (1½)

38. LHS $= I + A$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$ (1)

$$\text{RHS} = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}\right\}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1-0 & 0+\tan\alpha/2 \\ -\tan\alpha/2 & 1-0 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$ (1)

$$= \begin{bmatrix} \cos\alpha + \tan\frac{\alpha}{2}\sin\alpha & -\sin\alpha + \tan\frac{\alpha}{2}\cos\alpha \\ -\tan\frac{\alpha}{2}\cos\alpha + \sin\alpha & \tan\frac{\alpha}{2}\sin\alpha + \cos\alpha \end{bmatrix}$$

[multiplying rows by columns]

$$= \begin{bmatrix} \dfrac{\cos\alpha\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\sin\alpha}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\alpha\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\alpha}{\cos\frac{\alpha}{2}} \\ \dfrac{-\sin\frac{\alpha}{2}\cos\alpha + \sin\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\sin\frac{\alpha}{2}\sin\alpha + \cos\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$ (1)

$$= \begin{bmatrix} \dfrac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \dfrac{\sin\left(\frac{\alpha}{2} - \alpha\right)}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \dfrac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \end{bmatrix}$$ (1)

[$\because \cos(A - B) = \cos A\cos B + \sin A\sin B$ and $\sin(A - B) = \sin A\cos B - \cos A\sin B$]

$$= \begin{bmatrix} \dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$=$ LHS

Hence proved. (1)

Or

We have the following system of equations

$$x + y + z = 7000 \quad \ldots(i)$$

$$\Rightarrow \quad 10x + 16y + 17z = 110000 \quad \ldots(ii)$$

and

$$x - y = 0 \quad \ldots(iii)$$

This system of equations can be written in matrix from as $AX = B$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 10 & 16 & 17 \\ 1 & -1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7000 \\ 110000 \\ 0 \end{bmatrix}$ (1)

Here, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 16 & 17 \\ 1 & -1 & 0 \end{vmatrix}$

$$\Rightarrow \quad |A| = 1(0+17) - 1(0-17) + 1(-10-16)$$

$$= 17 + 17 - 26 = 8 \neq 0 \quad (1/2)$$

So, A is non-singular matrix and its inverse exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2 \begin{vmatrix} 16 & 17 \\ -1 & 0 \end{vmatrix} = 1(0+17) = 17$$

$$A_{12} = (-1)^3 \begin{vmatrix} 10 & 17 \\ 1 & 0 \end{vmatrix} = -1(0-17) = 17$$

$$A_{13} = (-1)^4 \begin{vmatrix} 10 & 16 \\ 1 & -1 \end{vmatrix} = 1(-10-16) = -26 \quad (1\tfrac{1}{2})$$

$$A_{21} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -1(0+1) = -1$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1(0-1) = -1$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1(-1-1) = 2$$

$$A_{31} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 16 & 17 \end{vmatrix} = 1(17-16) = 1$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 10 & 17 \end{vmatrix} = -1(17-10) = -7$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 10 & 16 \end{vmatrix} = 1(16-10) = 6 \quad (1)$$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 17 & 17 & -26 \\ -1 & -1 & 2 \\ 1 & -7 & 6 \end{bmatrix}^T = \begin{bmatrix} 17 & -1 & -1 \\ 17 & -1 & 7 \\ -26 & 2 & 6 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{\text{adj}(A)}{|A|} = \dfrac{1}{8}\begin{bmatrix} 17 & -1 & -1 \\ 17 & -1 & 7 \\ -26 & 2 & 6 \end{bmatrix}$ (1)

and the solution of given system is given by

$$X = A^{-1} B.$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 17 & -1 & -1 \\ 17 & -1 & 7 \\ -26 & 2 & 6 \end{bmatrix}\begin{bmatrix} 7000 \\ 110000 \\ 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 119000 - 110000 + 0 \\ 119000 - 110000 + 0 \\ -182000 + 220000 + 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 9000 \\ 9000 \\ 38000 \end{bmatrix} = \begin{bmatrix} 1125 \\ 1125 \\ 4750 \end{bmatrix} \quad (1)$$

On comparing the corresponding elements, we get $x = 1125$, $y = 1125$, $z = 4750$.