SOLUTIONS

1. Let $\text{cosec}^{-1}(-\sqrt{2}) = \theta$,

We know that, the principal value branch of cosec^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

$\therefore \quad \text{cosec}\,\theta = -\sqrt{2} = -\text{cosec}\,\frac{\pi}{4} = \text{cosec}\left(-\frac{\pi}{4}\right)$

$[\because \text{cosec}\,(-\theta) = -\text{cosec}\,\theta]$

$\Rightarrow \quad \theta = -\frac{\pi}{4}$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\Rightarrow \quad \text{cosec}^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$

Hence, the principal value branch of $\text{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

2. Let $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \theta$

$\Rightarrow \sin\theta = \frac{-1}{\sqrt{2}} \Rightarrow \sin\theta = -\sin\frac{\pi}{4} = \sin\left(-\frac{\pi}{4}\right)$

$\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \sin(-\theta) = -\sin\theta\right]$

$\because \quad \theta = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{-\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

3. Given, function is $f(x)=\dfrac{|x-1|}{x-1}, x \neq 1.$

The above function can be written as

$$f(x)=\begin{cases}\dfrac{x-1}{x-1}, & \text{if } x>1\\ -\dfrac{(x-1)}{x-1}, & \text{if } x<1\end{cases}$$

$\Rightarrow \quad f(x)=\begin{cases}1, & \text{if } x>1\\ -1, & \text{if } x<1\end{cases}$

Hence, the range of $f(x)$ is $\{-1, 1\}$.

4. transpose of matrix.

Or

$(fg-hc)$

$A_{21}=(-1)^{2+1}M_{21}=-M_{21}=-\begin{vmatrix}h & g\\ f & c\end{vmatrix}$

$=-(hc-fg)=fg-hc$

5. We have, $x\begin{bmatrix}2x & 2\\ 3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\ 4 & 4x\end{bmatrix}=2\begin{bmatrix}x^2+8 & 24\\ 10 & 6x\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2x^2 & 2x\\ 3x & x^2\end{bmatrix}+\begin{bmatrix}16 & 10x\\ 8 & 8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2x^2+16 & 12x\\ 3x+8 & x^2+8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$

$\Rightarrow \quad 12x=48$

$\therefore \quad x=4$

6. Given, $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$ is a skew-symmetric matrix.

$\therefore \quad A^T=-A$

$\Rightarrow \begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}^T=-\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}0 & 2 & b\\ a & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=\begin{bmatrix}0 & -a & 3\\ -2 & 0 & 1\\ -b & -1 & 0\end{bmatrix}$

On equating the corresponding elements, we get

$a=-2$ and $b=3$

Or

Given $B=\begin{bmatrix}2 & a & 5\\ -1 & 4 & b\\ c & -4 & 9\end{bmatrix}$ is a symmetric matrix.

$\therefore \quad A^T=A$

$\Rightarrow \begin{bmatrix}2 & -1 & c\\ a & 4 & -4\\ 5 & b & 9\end{bmatrix}=\begin{bmatrix}2 & a & 5\\ -1 & 4 & b\\ c & -4 & 9\end{bmatrix}$

$\Rightarrow \quad a=-1, b=-4, c=5$

$\therefore \quad a+b+c=-1-4+5=0$

7. We have, $f(x)=\sin x+\cos x$

$\therefore \quad f'(x)=\cos x-\sin x$

Now, for $x\in\left[0, \dfrac{\pi}{4}\right]$

$\cos x \geq \sin x$

$\Rightarrow \quad \cos x-\sin x\geq 0$

$\Rightarrow \quad f'(x)\geq 0$

So, $f(x)$ is increasing function.

Or

Given, $y=x^2e^{-x}$

$\Rightarrow \quad \dfrac{dy}{dx}=2xe^{-x}-x^2e^{-x}$

$\Rightarrow \quad \dfrac{dy}{dx}=xe^{-x}(2-x)$

For increasing function,

$\dfrac{dy}{dx}>0$

$\therefore \quad xe^{-x}(2-x)>0$

$\Rightarrow \quad xe^{-x}(x-2)<0$

$\therefore \quad x\in(0, 2)$

8. We have, $I=\int x^x(1+\log x)\,dx$

Put $x^x=t \Rightarrow x^x(1+\log x)\,dx=dt$

$\therefore \quad I=\int dt=t+c=x^x+c$

$=-e^x\cos x+c$

9. Given, function is $f(x)=|x+2|-1$

We know that, $|x+2|\geq 0$ for all $x\in R$.

Therefore, $f(x)=|x+2|-1\geq -1$ for every $x\in R$.

The minimum value of f is attained when $|x+2|=0$.

i.e. $|x+2|=0 \Rightarrow x=-2$

$\therefore$ Minimum value of $f=f(-2)$

$=|-2+2|-1$

$=0-1=-1$

Hence, $f(x)$ has minimum value -1 at $x=-2$, but $f(x)$ has no maximum value.

Or Given function is $h(x)=\sin(2x)+5$.

We know that,

$-1\leq \sin x\leq 1$

$\Rightarrow \quad -1\leq \sin 2x\leq 1$

$\Rightarrow \quad -1+5\leq \sin 2x+5\leq 1+5$

$\Rightarrow \quad 4\leq \sin 2x+5\leq 6$

Hence, maximum value of $h(x)$ is 6 and minimum value of $h(x)$ is 4.

10. It is given that

$$P(A)=\frac{1}{2} \text{ and } P(B)=0$$

$$\Rightarrow P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$

Hence, $P(B)=0$

$\therefore P\left(\frac{A}{B}\right)$ is not defined.

11. $0\le P(A/B)\le 1$

12. 0

Or

$$\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\hat{b}.$$

13. We know that, distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz+d=0$ is

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$\therefore$ Required distance

$$=\left|\frac{2+2(3)-2(-5)-9}{\sqrt{1^2+2^2+(-2)^2}}\right|=\left|\frac{2+6+10-9}{\sqrt{1+4+4}}\right|=\frac{9}{3}$$

$= 3$ units

14. Given, $2x-y+2z=5$

$\Rightarrow 2x-y+2z-5=0$ ...(i)

and $5x-2.5y+5z=20$

$\Rightarrow 5[x-0.5y+z]=20$

$\Rightarrow x-\frac{1}{2}y+z=4$

$\Rightarrow 2x-y+2z=8$

$\Rightarrow 2x-y+2z-8=0$ ...(ii)

Clearly, planes (i) and (ii) are parallel.

$\therefore$ Distance between two parallel planes,

$$d=\left|\frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}}\right|=\left|\frac{-8-(-5)}{\sqrt{(2)^2+(-1)^2+2^2}}\right|$$

$[\because d_2=-8, d_1=-5, a=2, b=-1 \text{ and } c=2]$

$$=\left|\frac{-8+5}{\sqrt{4+1+4}}\right|=\left|\frac{-3}{\sqrt{9}}\right|=|-1|=1$$

15. The equation of plane having intercepts 3, −4 and 2 is

$$\frac{x}{3}+\frac{y}{-4}+\frac{z}{2}=1$$

$\Rightarrow 4x-3y+6z=12$, which can be written as

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(4\hat{i}-3\hat{j}+6\hat{k})=12$$

$$\Rightarrow \vec{r}\cdot(4\hat{i}-3\hat{j}+6\hat{k})=12$$

which is the required vector equation of the given.

16. Given $\vec{a}=\vec{b}$

$\Rightarrow x\hat{i}+2\hat{j}-z\hat{k}=3\hat{i}-y\hat{j}+\hat{k}$

$\Rightarrow x=3, y=-2, z=-1$

$\therefore x+y+z=3-2-1=0$

17. The probability distribution of X is

X	0	1	2
$P(X)$	k	$2k$	$3k$

(i) (d) We know that

$\Sigma P(X)=1$

$\Rightarrow K+2K+3K=1$

$\Rightarrow 6K=1$

$\Rightarrow K=\frac{1}{6}$

(ii) (a) $P(X=2)=3K$

$=3\times\frac{1}{6}=\frac{1}{2}$

(iii) (d) $P(X>2)=0$

(iv) (a) $P(X<2)=P(X=0)+P(X=1)$

$=K+2K$

$=3K$

$=3\times\frac{1}{6}=\frac{1}{2}$

(v) (b) $P(0<x<2)=P(X=1)$

$=2K$

$=2\times\frac{1}{6}=\frac{1}{3}$.

18. (i) (b) We have

$$S=2\left(x\times 2x+2x\times\frac{x}{3}+\frac{x}{3}\times x\right)+4\pi y^2$$

$$\Rightarrow S=6x^2+4\pi y^2$$

(ii) (a) We have,

$$V=\frac{4}{3}\pi y^3+x\times 2x\times\frac{x}{3}$$

$$\Rightarrow V=\frac{4}{3}\pi y^3+\frac{2}{3}x^3$$

(iii) We have (from part (ii))

$$V=\frac{4}{3}\pi y^3+\frac{2}{3}x^3$$

$$=\frac{4}{3}\pi\left(\frac{S-6x}{4\pi}\right)^{\frac{3}{2}}+\frac{2}{3}x^3$$

$$\therefore \frac{dV}{dx}=\frac{1}{6\sqrt{\pi}}\times\frac{3}{2}(S-6x^2)^{\frac{1}{2}}(-12x)+\frac{2}{3}\times 3x^2$$

$$=-\frac{3}{\sqrt{\pi}}(S-6x^2)^{\frac{1}{2}}x+2x^2$$

For minimum, $\frac{dV}{dx} = 0$

$$\Rightarrow \quad -\frac{3}{\sqrt{\pi}}(S-6x^2)^{\frac{1}{2}}x + 2x^2 = 0$$

$$\Rightarrow \quad 2x^2 = \frac{3x}{\sqrt{\pi}}(S-6x^2)^{\frac{1}{2}}$$

$$\Rightarrow \quad 2x^2 = \frac{3x}{\sqrt{\pi}}(4\pi y^2)^{\frac{1}{2}}$$

$$\Rightarrow \quad 2\sqrt{\pi}x = 3(4\pi y^2)^{\frac{1}{2}}$$

$$\Rightarrow \quad 4\pi x^2 = 9 \times 4\pi y^2$$

$$\Rightarrow \quad x^2 = 9y^2$$

$$\Rightarrow \quad x = 3y$$

(iv) Minimum value of $V = \frac{4}{3}\pi y^3 + \frac{2}{3}x^3$

$$= \frac{4}{3}\pi\left(\frac{x}{3}\right)^3 + \frac{2}{3}x^3$$

$$= \frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$$

(v) (a) When V is minimum, $S = 6x^2 + 4\pi y^2$

$$= 6x^2 + 4\pi\left(\frac{x}{3}\right)^2$$

$$= 6x^2 + \frac{4}{9}\pi x^2$$

$$= 2x^2\left[3+\frac{2}{9}\pi\right]$$

19. We have, $y = |x - x^2| = \begin{cases} x - x^2, & \text{if } 0 \le x \le 1 \\ x^2 - x, & \text{if } x > 1 \end{cases}$ (1/2)

At $x = 1$,

$$\text{LHD} = \left[\frac{d}{dx}(x - x^2)\right]_{\text{at } x=1} = 1 - 2 = -1$$ (1)

$$\text{RHD} = \left[\frac{d}{dx}(x^2 - x)\right]_{\text{at } x=1} = 2 - 1 = 1$$

$\therefore \quad \text{LHD} \ne \text{RHD}$

Hence, $\frac{dy}{dx}$ at $x = 1$ does not exist. (1/2)

20. We have, $x = 1 - a\sin\theta$ ...(i)

and $y = b\cos^2\theta$ ...(ii)

On differentiating Eqs. (i) and (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = -a\cos\theta \text{ and } \frac{dy}{d\theta} = -2b\cos\theta\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta} = \frac{2b\sin\theta}{a}$$ (1)

Hence, required slope of normal

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } \theta = \frac{\pi}{2}}} = \frac{-1}{\frac{2b}{a}\sin\left(\frac{\pi}{2}\right)} = \frac{-a}{2b}$$ (1)

Or

We have, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$

$$\Rightarrow \quad f'(x) = \sqrt{3}\cos x + \sin x - 2a$$

$$= 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right) - 2a$$

$$= 2\left(\cos\frac{\pi}{6}\cos x + \sin\frac{\pi}{6}\sin x\right) - 2a$$ (1)

$$= 2\cos\left(\frac{\pi}{6} - x\right) - 2a$$

$$= 2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right] \le 0, \forall x \in R$$

$$\left[\because \cos\left(\frac{\pi}{6} - x\right) \le 1 \text{ and } a \ge 1\right]$$

$\therefore f(x)$ is decreasing on R. (1)

21. Let A be a symmetric matrix and $n \in N$. Then,

$$A^n = AAA \ldots A \text{ upto } n\text{-times}$$

$$\Rightarrow \quad (A^n)^T = (AAA \ldots A \text{ upto } n\text{-times})^T$$ (1)

$$\Rightarrow \quad (A^n)^T = (A^T A^T A^T \ldots A^T \text{ upto } n\text{-times})$$

$$\Rightarrow \quad (A^n)^T = (A^T)^n = A^n \quad [\because A^T = A]$$

Hence, A^n is also a symmetric matrix. (1)

Or

We have,

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\therefore \quad |A| = 3(3-0) - 0(2-0) - 1(8-0)$$

$$= 9 - 8$$

$$= 1$$ (1)

Now, we know that

$$|\text{adj (adj A)}| = |A|^{(n-1)^2}$$

$\therefore$ For given matrix

$$|\text{adj (adj A)}| = |A|^{(3-1)^2} = (1)^4 = 1$$ (1)

22. The position vectors of the points A, B and C are $\overrightarrow{OA} = \hat{i} - 2\hat{j} - 8\hat{k}$, $\overrightarrow{OB} = 5\hat{i} - 2\hat{k}$ and $\overrightarrow{OC} = 11\hat{i} + 3\hat{j} + 7\hat{k}$ respectively.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\hat{i} - 2\hat{k}) - (\hat{i} - 2\hat{j} - 8\hat{k})$$

$$= 4\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (11\hat{i} + 3\hat{j} + 7\hat{k}) - (5\hat{i} - 2\hat{k})$$

$$= 6\hat{i} + 3\hat{j} + 9\hat{k}$$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (11\hat{i} + 3\hat{j} + 7\hat{k}) - (i - 2\hat{j} - 8\hat{k})$

$= 10\hat{i} + 5\hat{j} + 15\hat{k}$

Here, we see that $\overrightarrow{AB} = 2(2\hat{i} + \hat{j} + 3\hat{k}) = \frac{2}{5}\overrightarrow{AC}$ (1)

Thus, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel.
But AB and AC have a common point.
So, the points A, B and C are collinear.

Now, $\overrightarrow{AB} = 2(2\hat{i} + \hat{j} + 3\hat{k}) = \frac{2}{3}\overrightarrow{BC}$

$\Rightarrow \quad |\overrightarrow{AB}| = \frac{2}{3}|\overrightarrow{BC}| \Rightarrow AB = \frac{2}{3}BC \Rightarrow \frac{AB}{BC} = \frac{2}{3}$

Hence, B divides the line segment AC internally in the ratio $2:3$. (1)

23. We have, $\tan^{-1}\sqrt{x^2 + x} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$

This equation holds, if

$x^2 + x \geq 0$ and $0 \leq x^2 + x + 1 \leq 1$

Now, $x^2 + x \geq 0$ and $0 \leq x^2 + x + 1 \leq 1$

$\Rightarrow \quad x^2 + x \geq 0$ and $x^2 + x + 1 \leq 1$

$[\because x^2 + x + 1 > 0$ for all $x]$ (1)

$\Rightarrow \quad x^2 + x \geq 0$ and $x^2 + x \leq 0$

$\Rightarrow \quad x^2 + x = 0$

$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0, -1$

Clearly, these two values satisfy the given equation. Hence, $x = 0, -1$ are the solution of the given equation. (1)

24. For a pair of dice, total number of events,

$n(S) = 6 \times 6 = 36$

Number of doublets = 6
which are $\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

Let $R = P$ (doublets in pair of dice) $= \frac{6}{36} = \frac{1}{6}$

and $Q = P$ (non-doublets in pair of dice) $= 1 - \frac{1}{6} = \frac{5}{6}$ (1/2)

Let X denotes the number of doublets in pair of dice.
Then, X can take values 0, 1, 2, 3.

$P(X = 0) = P$ (no doublet)

$= Q^3 = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$ (1/2)

$P(X = 1) = P$ (one doublet and two non-doublets)

$= QQR + RQQ + QRQ = 3Q^2R$

$= 3 \times \left(\frac{5}{6}\right)^2 \times \left(\frac{1}{6}\right) = 3 \times \frac{25}{36} \times \frac{1}{6} = \frac{75}{216}$

$P(X = 2) = P$ (two doublets and one non-doublet)

$= RRQ + RQR + QRR$

$= 3QR^2 = 3 \times \frac{5}{6} \times \left(\frac{1}{6}\right)^2$

$= 3 \times \frac{5}{6} \times \frac{1}{36} = \frac{15}{216}$ (1/2)

and $P(X = 3) = P$ (three doublets)

$= R^3 = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$

$\therefore$ Required probability distribution is

X	0	1	2	3
$P(X)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

(1/2)

25. We have, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$

$= \hat{i}(-2 - 15) - \hat{j}(-4 - 9) + \hat{k}(10 - 3)$ (1)

$= -17\hat{i} + 13\hat{j} + 7\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2}$

$= \sqrt{289 + 169 + 49}$

$= \sqrt{507}$ (1)

26. Let $I = \int \frac{1}{\sqrt{9 + 8x - x^2}}\,dx$

$= \int \frac{1}{\sqrt{-[x^2 - 8x - 9]}}\,dx$ (1/2)

$= \int \frac{1}{\sqrt{-[(x-4)^2 - 16 - 9]}}\,dx$

$= \int \frac{1}{\sqrt{25 - (x-4)^2}}\,dx$

$= \int \frac{1}{\sqrt{(5)^2 - (x-4)^2}}$

$= \sin^{-1}\frac{x-4}{5} + C$ (1½)

Or

Let $I = \int x \log x$

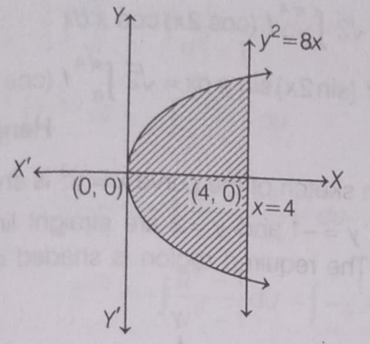
$= \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$ (1/2)

$= \frac{1}{2}x^2 \log x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2 \log x - \frac{x^2}{4} + C$

$= \frac{x^2}{2}\left[\log x - \frac{1}{2}\right] + C$ (1½)

27.



Required area $= 2\int_0^4 y\,dx$

$$= 2\int_0^4 \sqrt{8x}\,dx \quad (1)$$

$$= 4\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= \frac{8\sqrt{2}}{3}[2\sqrt{2} - 0]$$

$$= \frac{32}{3} \text{ sq unit} \quad (1)$$

28. Let $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$ ...(i)

$$\Rightarrow I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$\Rightarrow I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx \quad \text{...(ii) (1)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_1^2 1\,dx$$

$$\Rightarrow 2I = [x]_1^2$$

$$\Rightarrow 2I = 2 - 1 = 1$$

$$\Rightarrow I = \frac{1}{2} \quad (1)$$

29. Given, $f: R \to R$ defined by $f(x) = 2x^3 - 5$

For one-one (injective)

Let $f(x_1) = f(x_2)$, $\forall\, x_1, x_2 \in R$

$$\Rightarrow 2x_1^3 - 5 = 2x_2^3 - 5$$

$$\Rightarrow 2x_1^3 = 2x_2^3$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow x_1 = x_2$$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

So, f is one-one (injective). (1)

For onto (surjective)

Let y be an arbitrary element of R (codomain), then

$$f(x) = y$$

$$\Rightarrow 2x^3 - 5 = y$$

$$\Rightarrow 2x^3 = y + 5$$

$$\Rightarrow x^3 = \frac{y+5}{2}$$

$$\Rightarrow x = \left(\frac{y+5}{2}\right)^{1/3} \quad (1)$$

Clearly, $x \in R$ (domain), $\forall\, y \in R$ (codomain).

Thus, for each $y \in R$ (codomain) there exists

$x = \left(\frac{y+5}{2}\right)^{1/3} \in R$ (domain) such that

$$f(x) = f\left[\left(\frac{y+5}{2}\right)^{1/3}\right] = 2\left\{\left(\frac{y+5}{2}\right)^{1/3}\right\}^3 - 5$$

$$= 2\left\{\left(\frac{y+5}{2}\right)\right\} - 5 = y + 5 - 5 = y$$

This shows that every element in the codomain has its pre-image in the domain.

So, f is onto (or f is surjective).

Thus, f is both one-one and onto (or both injective and surjective). Hence, f is bijective. **Hence proved.** (1)

30. Let $I = \int \frac{1}{3x^2+5x+7}\,dx = \int \frac{dx}{3\left(x^2 + \frac{5x}{3} + \frac{7}{3}\right)}$

$$= \frac{1}{3}\int \frac{dx}{x^2 + \frac{5x}{3} + \frac{7}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{x^2 + \frac{5x}{3} + \frac{7}{3} + \frac{25}{36} - \frac{25}{36}} \quad (1)$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{5}{6}\right)^2 + \left(\frac{7}{3} - \frac{25}{36}\right)}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{5}{6}\right)^2 + \left(\frac{84-25}{36}\right)}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2}$$

$$= \frac{1}{3}\cdot\frac{6}{\sqrt{59}}\tan^{-1}\left(\frac{x+\frac{5}{6}}{\frac{\sqrt{59}}{6}}\right) + c \quad (1)$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\therefore I = \frac{2}{\sqrt{59}}\tan^{-1}\left(\frac{6x+5}{\sqrt{59}}\right) + C \quad (1)$$

Or

Let $I = \int \frac{xe^x}{(x+1)^2} dx = \int \frac{(x+1-1)e^x}{(x+1)^2} dx$ (1)

$= \int \left[\frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}\right] e^x dx$

$= \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x dx$

Now, consider $f(x) = \frac{1}{1+x}$, then $f'(x) = \frac{-1}{(1+x)^2}$. (1)

Thus, the given integrand is of the form

$$\int e^x (f(x) + f'(x))\, dx.$$

Hence, $I = \frac{e^x}{x+1} + c$ $\left[\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x)\right]$ (1)

31. Let $I = \int_0^{\pi/2} f(\sin 2x) \sin x\, dx$ ...(i)

$\Rightarrow I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2} - x\right)\right\} \sin\left(\frac{\pi}{2} - x\right) dx$

$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$

$\Rightarrow I = \int_0^{\pi/2} f\{\sin(\pi - 2x)\} \cos x\, dx$

$\Rightarrow I = \int_0^{\pi/2} f(\sin 2x) \cos x\, dx$ ...(ii) (1)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/2} f(\sin 2x) \cdot (\sin x + \cos x)\, dx$

$\Rightarrow 2I = 2\int_0^{\pi/4} f(\sin 2x) \cdot (\sin x + \cos x)\, dx$

$\left[\because \int_0^{2a} f(x)\, dx = \int_0^a 2f(x)\, dx, \text{if } f(2a-x) = f(x)\right]$

$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f(\sin 2x)\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right) dx$

$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f(\sin 2x)$
$\left(\cos\frac{\pi}{4} \sin x + \sin\frac{\pi}{4} \cos x\right) dx$

$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f(\sin 2x) \sin\left(x + \frac{\pi}{4}\right) dx$

$[\because \sin A \cos B + \cos A \sin B = \sin(A+B)]$ (1)

$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f\left\{\sin 2\left(\frac{\pi}{4} - x\right)\right\} \sin\left(\frac{\pi}{4} - x + \frac{\pi}{4}\right) dx$

$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$

$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f\left\{\sin\left(\frac{\pi}{2} - 2x\right)\right\} \sin\left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x\, dx$

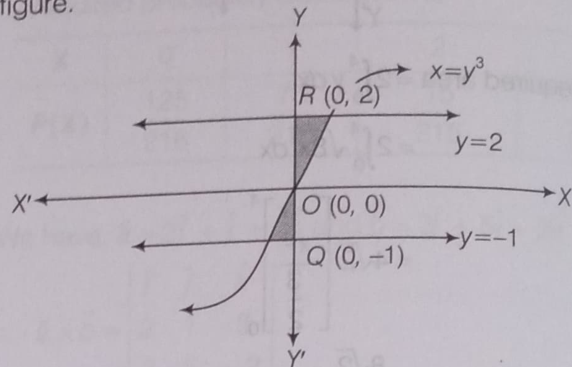
$\left[\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$

$\Rightarrow I = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x\, dx$

$\therefore \int_0^{\pi/2} f(\sin 2x) \sin x\, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x\, dx$

Hence proved. (1)

32. A rough sketch of the curve $x = y^3$ is shown below. Clearly, $y = -1$ and $y = 2$ are straight lines parallel to X-axis. The required region is shaded in given below figure. (1)



(1)

So, required area A is given by

$A = \int_{-1}^{2} |x|\, dy = \int_{-1}^{0} |x|\, dy + \int_0^2 |x|\, dy$

$= \int_{-1}^{0} -x\, dy + \int_0^2 x\, dy$

$= \int_{-1}^{0} -y^3 dy + \int_0^2 y^3 dy$

$= -\left[\frac{y^4}{4}\right]_{-1}^{0} + \left[\frac{y^4}{4}\right]_0^2$

$= -\left[0 - \frac{(-1)^4}{4}\right] + \left[\frac{2^4}{4} - 0\right]$

$= \frac{1}{4} + 4 = \frac{17}{4}$ sq unit. (1)

33. Given differential equation is

$$\sqrt{1 + x^2 + y^2 + x^2 y^2} + xy\frac{dy}{dx} = 0$$

$\Rightarrow \sqrt{1(1+x^2) + y^2(1+x^2)} = -xy\frac{dy}{dx}$

$\Rightarrow \sqrt{(1+x^2)(1+y^2)} = -xy\frac{dy}{dx}$

$\Rightarrow \sqrt{1+x^2} \cdot \sqrt{1+y^2} = -xy\frac{dy}{dx}$

$\Rightarrow \frac{y}{\sqrt{1+y^2}} dy = -\frac{\sqrt{1+x^2}}{x} dx$ (1)

On integrating both sides, we get

$$\int \frac{y}{\sqrt{1+y^2}} dy = -\int \frac{\sqrt{1+x^2}}{x^2} \cdot x\, dx$$

On putting $1 + y^2 = t$ and $1 + x^2 = u^2$

$\Rightarrow 2y\, dy = dt$ and $2x\, dx = 2u\, du$

$\Rightarrow y\, dy = \frac{dt}{2}$ and $x\, dx = u\, du$

$\therefore \quad \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\int \frac{u}{u^2-1} \cdot u\, du$

$\Rightarrow \quad \frac{1}{2}\int t^{-1/2}\, dt = -\int \frac{u^2}{u^2-1}\, du$

$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} = -\int \frac{(u^2-1+1)}{u^2-1}\, du$ (1)

$\Rightarrow \quad t^{1/2} = -\int \frac{u^2-1}{u^2-1}\, du - \int \frac{1}{u^2-1}\, du$

$\Rightarrow \quad \sqrt{1+y^2} = -\int du - \int \frac{1}{u^2-(1)^2}\, du$

[put $1+y^2=t$]

$\Rightarrow \sqrt{1+y^2} = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C$

$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

$\therefore \quad \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$

which is the required solution. (1)

34. Given, $y = e^{a\sin^{-1}x}$ ...(i)

On differentiating both sides of Eq. (i), we get

$\frac{dy}{dx} = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}}\, y$ (1)

$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = ay$ ...(ii)

On differentiating both sides of Eq. (ii), we get

$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{1}{2\sqrt{1-x^2}}(-2x)\frac{dy}{dx} = a\frac{dy}{dx}$ (1)

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a\sqrt{1-x^2}\frac{dy}{dx}$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a(ay)$ [from Eq. (ii)]

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$ **Hence proved.** (1)

Or

We have, $y = e^x \sin x$ ...(i)

On differentiating both the sides of Eq. (i), we get

$\frac{dy}{dx} = e^x \cos x + \sin x\, e^x$

$\Rightarrow \quad \frac{dy}{dx} = e^x(\cos x + \sin x)$ ...(ii) (1)

Again, differentiating both sides w.r.t. x, we get

$\frac{d^2y}{dx^2} = e^x(-\sin x + \cos x) + e^x(\cos x + \sin x)$

$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos x e^x$ ...(iii)

Now,

LHS $= \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$ (1)

$= 2\cos x e^x - 2[e^x(\cos x + \sin x)] + 2[e^x \sin x]$

$= 2e^x\cos x - 2e^x\cos x - 2e^x\sin x + 2e^x \sin x$

$= 0$

$=$ RHS **Hence proved.** (1)

35. Given, $f(x) = \begin{cases} 2x-1, & x<2 \\ a, & x=2 \\ x+1, & x>2 \end{cases}$ is continuous at $x=2$

$\therefore (\text{LHL})_{x=2} = (\text{RHL})_{x=2} = f(2)$ ...(i)

Now, $f(2) = a$ (1)

and LHL $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x-1)$

$= \lim_{h\to 0} [2(2-h)-1] = 3$

$\therefore \quad$ LHL $= f(2)$ (1)

$\Rightarrow \quad a = 3$

Now, let us check the continuity at $x = 3$.

Consider $\lim_{x\to 3} f(x) = \lim_{x\to 3} (x+1)$ $[\because f(x) = x+1, x>2]$

$= 4 = f(3)$ $[\because f(3) = 3+1 = 4]$

$\therefore f(x)$ is continuous at $x = 3$. (1)

36. $\therefore$ The given LPP is

Maximise $Z = 24x + 18y$

Subject to constraints

$2x+3y \le 10, 3x+2y \le 10, x \ge 0, y \ge 0.$

Let us consider the inequalities as equation,

$2x + 3y = 10$ ...(i)

and $3x + 2y = 10$ ...(ii)

Table for line $2x+3y=10$ is

X	0	55
y	10/3	0

(1)

So, it passes through (0, 10/3) and (5, 0).

On putting (0, 0) in the inequality $2x+3 \le 10$,

we get $2(0)+3(0) \le 10$

$\Rightarrow \quad 0 \le 10$ (which is true)

So, the half plane is towards the origin. (1)

Table for line $3x+2y=10$ is

X	0	10/3
y	5	0

So, it passes through (0, 5) and (10/3, 0)

On putting (0, 0) in the inequality $3x+2y \le 0$,

we get $3(0)+2(0) \le 10$

$\Rightarrow \quad 0 \le 10$ (which is true)

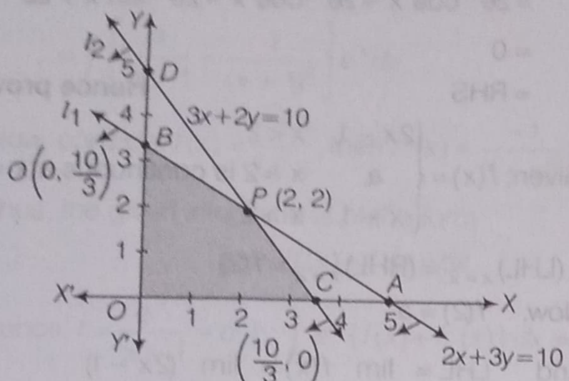
So, the half plane is towards the origin.

On solving Eq. (i) and Eq. (ii), we get

$$x = 2 \text{ and } y = 2$$

So, the intersection point is (2, 2). (1)

The graphical representation of the system of inequation is given below



(1)

The feasible region is OCPB and the corner points are $O(0,0)$, $B\left(0, \frac{10}{3}\right)$, $C\left(\frac{10}{3}, 0\right)$ and $P(2,2)$.

Corner points	Value of $Z = 24x + 18y$
$O(0,0)$	0
$C\left(\frac{10}{3}, 0\right)$	80
$P(2,2)$	84 (Maximum)
$B\left(0, \frac{10}{3}\right)$	60

Thus, profit will be maximum, when 2 packages of nuts and 2 packages of bolts are manufactured. (1)

Or

The given LPP is

Maximise $Z = 100x + 120y$

Subject to constraints are

$$2x + 3y \le 30 \quad \ldots(i)$$

$$3x + y \le 17 \quad \ldots(ii)$$

and $$x, y \ge 0$$

Consider the inequalities as equations, we get

$$2x + 3y = 30 \quad \ldots(iii)$$

and $$3x + y = 17 \quad \ldots(iv)$$

Table for line $2x + 3y = 30$ is

x	0	15
y	10	0

$\therefore$ Eq. (iii) passes through points (0,10) and (15,0).

On putting (0, 0) in the inequality $2x + 3y \le 30$, we get

$$2(0) + 3(0) \le 30$$

$\Rightarrow$ $$0 \le 30$$ [true] (1)

$\therefore$ The shaded portion is towards the origin.

Table for line $3x + y = 17$ is

x	0	17/3
y	17	0

$\therefore$ Eq. (iv) passes through the points (0,17) and (17/3, 0). (1)

On putting (0, 0) in the inequality $3x + y \le 17$, we get

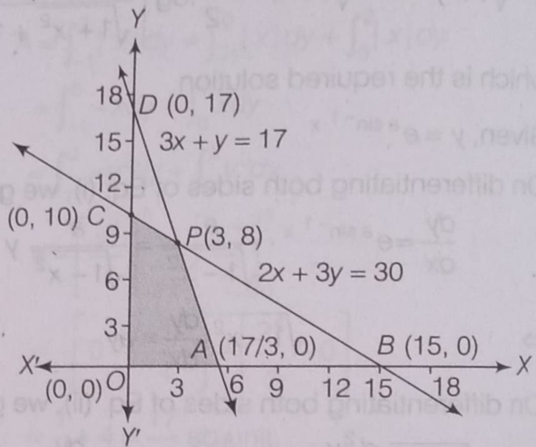
$$3(0) + 0 \le 17$$

$\Rightarrow$ $$0 \le 17$$ [true]

$\therefore$ The shaded portion is towards the origin.

The intersection point of both lines is $P(3, 8)$.

Now, on plotting these points on graph paper, we get the feasible, region $OAPCO$, which is bounded and its corner points are $O(0,0)$, $A\left(\frac{17}{3}, 0\right)$, $P(3, 8)$ and $C(0,10)$. (1)



(1)

The values of Z at corner points are given below

Corner points	$Z = 100x + 120y$
$O(0,0)$	$100 \times 0 + 120 \times 0 = 0 + 0 = 0$
$A\left(\frac{17}{3}, 0\right)$	$100 \times \frac{17}{3} + 120 \times 0 = \frac{1700}{3} = 566.66$
$C(0,10)$	$100 \times 0 + 120 \times 10 = 0 + 1200 = 1200$
$P(3,8)$	$100 \times 3 + 120 \times 8 = 300 + 960 = 1260$ [maximum]

Here, the maximum value of Z is 1260 at point $P(3, 8)$.

(1)

37. Let the first, second and third number be x, y and z respectively. Then, according to given conditions, we have

$$x + y + z = 6, \; y + 3z = 11$$

and $$x + z = 2y \text{ or } x - 2y + z = 0$$

This system of equation can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

or $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix} \quad (1)$$

Now, consider $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

$$= 1(1+6) - 1(0-3) + 1(0-1)$$
$$= 7 + 3 - 1 = 9 \neq 0$$

$\therefore A^{-1}$ exists. (1)

Now, let us find the cofactor of elements of $|A|$.

Clearly, $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7$, $C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 3$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$, $C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -3$,

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, $C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$,

$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$, $C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3$

and $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ (1)

$$\therefore \text{adj}(A) = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

and $A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$ (1)

Now, as $X = A^{-1}B$ $[\because AX = B]$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(1)

Hence, $x = 1$, $y = 2$ and $z = 3$.

Or

Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

To prove $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in N$

Proof We shall prove this statement by using principle of mathematical induction.

Now, let $P(n)$ be the given statement,

i.e. $P(n): A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ (1)

For $n = 1$, we have

$$P(1): A^1 = \begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A$$

$\therefore$ Statement is true for $n = 1$. (1)

Now, let us assume that statement is true for $n = k$, we have

$$P(k): A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix} \quad \text{...(i)}$$

(1)

Now, we shall prove the statement for $n = k + 1$, we have to show, $P(k+1): A^{k+1} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$

Consider, LHS $= A^{k+1} = A^k \cdot A$

$$= \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \\ 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \\ 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \end{bmatrix} \quad \text{[using Eq. (i)]}$$

$$= \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix} \quad [\because 3 \cdot 3^{k-1} = 3^{1+k-1} = 3^k]$$

$=$ RHS

$\therefore$ Statement is true for $n = k + 1$.

Hence, by principle of mathematical induction, statement is true for all n, where $n \in N$. (1)

38. Let the equation of plane in intercept form be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0 \quad \text{...(i) (1/2)}$$

It cut the axes on the point $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the centroid of ΔABC. (1/2)

Then,

$$\alpha=\frac{a+0+0}{3}, \beta=\frac{0+b+0}{3} \text{ and } \gamma=\frac{0+0+c}{3}$$

$\Rightarrow \quad a=3\alpha$, $b=3\beta$ and $c=3\gamma$ (1)

Now, the perpendicular distance from origin (0, 0, 0) to the plane (i) is

$$3p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right| \quad (1/2)$$

$$\Rightarrow \quad 3p=\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| \quad (1/2)$$

$$\Rightarrow \quad \frac{1}{3p}=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

On squaring both sides, we get

$$\frac{1}{9p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad (1/2)$$

On putting the values of a, b and c, we get

$$\frac{1}{9p^2}=\frac{1}{(3\alpha)^2}+\frac{1}{(3\beta)^2}+\frac{1}{(3\gamma)^2} \quad (1/2)$$

$$\Rightarrow \quad \frac{1}{9p^2}=\frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2}$$

$$\Rightarrow \quad \frac{1}{9p^2}=\frac{1}{9}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)$$

$$\Rightarrow \quad \frac{1}{p^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} \quad (1/2)$$

Hence, the locus of the centroid (α, β, γ) is

$$\frac{1}{p^2}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \quad [\text{replace } \alpha, \beta, \gamma \text{ by } x, y, z]$$

$\therefore \quad p^{-2}=x^{-2}+y^{-2}+z^{-2}$ **Hence proved.** (1/2)

Or

Given equation of lines are

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4} \quad \text{...(i)}$$

and

$$\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1} \quad \text{...(ii)}$$

Since, these lines intersect therefore the shortest distance between them will be zero.

Now, comparing these lines with

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, we get

$$x_1=1, \quad y_1=-1, \quad z_1=1$$
$$x_2=3, \quad y_2=k, \quad z_2=0 \quad (1)$$
$$a_1=2, \quad b_1=3, \quad c_1=4$$
$$a_2=1, \quad b_2=2, \quad c_2=1$$

We know that if two lines intersect, then shortest distance between them $=0$

$$\therefore \quad \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

$$\therefore \quad \begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad 2(3-8)-(k+1)(2-4)-1(4-3)=0$$

$$\Rightarrow \quad 2(-5)-(k+1)(-2)-1(1)=0 \quad (1)$$

$$\Rightarrow \quad -10+2(k+1)-1=0$$

$$\Rightarrow \quad 2(k+1)=11$$

$$\Rightarrow \quad k=\frac{11}{2}-1=\frac{9}{2} \quad (1)$$

Now, let the required equation of plane be

$$A(x-1)+B(y+1)+C(z-1)=0$$

where, A, B and C are DR's ratios of normal and $(1, -1, 1)$ is the point on the line (i). [$\because$ equation of plane is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$]

As we know that, the normal to the plane is also a normal to the lines lying in that plane, therefore

$$2A+3B+4C=0$$

and $A+2B+C=0$ [$\because A_1A_2+B_1B_2+C_1C_2=0$] (1)

On solving above equations, we get

$$\frac{A}{3-8}=\frac{-B}{2-4}=\frac{C}{4-3}$$

$$\Rightarrow \quad \frac{A}{-5}=\frac{-B}{-2}=\frac{C}{1} \Rightarrow \frac{A}{-5}=\frac{B}{2}=\frac{C}{1}$$

Thus, the required equation of the plane is

$$-5(x-1)+2(y+1)+1(z-1)=0$$

$$\Rightarrow \quad -5x+5+2y+2+z-1=0$$

$$\Rightarrow \quad 5x-5-2y-2-z+1=0$$

$$\Rightarrow \quad 5x-2y-z=6 \quad (1)$$